ON FREENESS OF COMPACTLY INDUCED MOD-p REPRESENTATIONS OF $\mathrm{SL}_2(\mathrm{F})$

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Abstract. Let p be a prime, and F a non-archimedean local field with residue characteristic p and ring of integers $\mathcal{O}_{\mathrm{F}}$. Set $\mathrm{G_S} := \mathrm{SL}_2(\mathrm{F})$ and $\mathrm{K}_0 := \mathrm{SL}_2(\mathcal{O}_{\mathrm{F}})$. For a smooth irreducible $\bar{\mathbf{F}}_p$-representation σ of K_0, we study the structure of the compact induction $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma)$ as a left module over the standard spherical Hecke algebra $\mathrm{End}_{\mathrm{G_S}}\left(\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma)\right)$. We prove that it is free and of infinite rank.

1. Introduction

The study of smooth mod-p representations of p-adic groups has been a central theme in the mod-p and p-adic Langlands programs. Following the pioneering work of Barthel-Livné [5] on GL_2, much effort has gone into understanding the structure and classification of such representations, particularly the supersingular ones. Among the four types – characters, principal series, Steinberg, and supersingular – much is known about the first three, but supersingular representations remain more mysterious in general. For $\mathrm{GL}_2(\mathbf{Q}_p)$, Breuil [6] achieved a complete classification by constructing explicit models, but for general local fields and for groups of higher ranks, the picture remains incomplete and technically subtle. However, the analogue of Barthel–Livné's classification for the group $\mathrm{SL}_2(\mathrm{F})$ in the mod-p setting was subsequently established in [1, 7]. Also, Herzig extended this classification for p-adic GL_n in his pioneering work [10, 11]. More recently, this classification has been further generalized to connected reductive p-adic groups through the far-reaching work of Abe-Henniart-Herzig-Vignéras (see [2, 3]).

In all of the above works the structure of the compactly induced representations and their pro-p-Iwahori invariants plays a crucial role. For example, if G is split reductive p-adic group and σ is a weight of some hyperspecial maximal compact subgroup K then $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}(\sigma)$ is torsion free as a module over $\mathcal{H}(\mathrm{G}, \mathrm{K}, \sigma) := \mathrm{End}_{\mathrm{G}}(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}(\sigma))$ (see [10, Corollary 6.5]). When G has semisimple rank 1 and is split or quasi-split, say $\mathrm{GL}_2, \mathrm{SL}_2$ or the unramified U(2,1), finer structural results are known. For instance, in the above three examples the explicit right action of the pro-p-Iwahori Hecke algebra on the pro-p-Iwahori invariants of the compactly induced representations can be computed (see [5] for GL_2, [8] for SL_2, and [15] for U(2,1)).

In the present paper we consider the p-adic group SL_2 and we show that the compactly induced representations as left modules over their spherical Hecke algebras are free and of infinite rank, thus refining [10, Corollary 6.5] in this special case. Our main theorem is as follows.

Theorem 1.1 (Theorem 4.3)**.** — *The compactly induced representation* $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$ *is a free module of infinite rank over the spherical Hecke algebra* $\mathcal{H}(\mathrm{G_S}, \mathrm{K}_0, \sigma_{\vec{r}})$.

Our result complements the analogous freeness results previously established for GL_2 (see [5, Section 5]) and for the unramified unitary group U(2,1) (see [14]). There are, however, two notable distinctions. First, we isolate and formalize the key combinatorial–linear algebraic argument that underlies both proofs in [5, 14]; this is stated abstractly as Lemma 3.1. This conceptual clarification separates the structural core of the argument from the group-specific computations, making the mechanism of freeness more transparent and potentially applicable to other p-adic groups. Second, we give an explicit proof of the non-vanishing of the standard Hecke operator on the subspace

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of functions supported on the central vertex of the Bruhat–Tits tree of SL_2; this is Lemma 4.5. Although this fact is expected on general grounds, its verification requires a careful analysis of the Hecke action and explicit realization of mod p irreducibles of the finite group $\mathrm{SL}_2(\mathbf{F}_q)$ for a p-power q. This, to our knowledge, has not appeared explicitly in the literature. It was used implicitly, for example, in the proof of the main theorem of [14].

Finally, we note that Lemma 4.6, together with [8, Propositions 4.9 and 4.10], provides a detailed description of the explicit Hecke actions on the pro-p-Iwahori invariants of compactly induced representations of SL_2 in the mod p setting. The significance of this analysis is underscored both by the main theorem of the present paper and by the existence of Hecke eigenvalues for SL_2 (see [8, Proposition 4.14]), analogous to the GL_2 case [5, Proposition 32]; see also [4, Question 8].

Outline of the proof of the main theorem. – The proof proceeds in two stages.

(1) In section 3 we prove a general combinatorial–linear algebra assertion : if a vector space V over any field admits a graded decomposition $\mathrm{V} = \bigoplus_{k\geq 0} \mathrm{C}_k$ and there is a linear operator T satisfying certain *injectivity*, *triangularity*, and *filtration compatibity* conditions, then we can choose subsets $\mathrm{A}_k \subset \mathrm{C}_k$ inductively so that the family $\{\mathrm{T}^i(\mathrm{A}_j) \,|\, 0 \leq i, j \leq n,\ i + j \leq n\}$ of sets are mutually disjoint and their union forms a basis of the truncated sum $\mathrm{B}_n := \bigoplus_{k\leq n} \mathrm{C}_k$.
This yields an explicit combinatorial construction of a free T-module basis $\bigsqcup_{k\geq 0} \mathrm{A}_k$ of V.

(2) In section 4 we apply this framework to $\mathrm{V} = \mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G}_\mathrm{S}}(\sigma_{\vec{r}})$ and $\mathrm{T} = \tau$. The relevant subspaces C_k are obtained from the Cartan-Iwahori decomposition

$$\mathrm{G}_\mathrm{S} = \bigsqcup_{n\in\mathbf{Z}} \mathrm{K}_0 \alpha_0^n \mathrm{I}_\mathrm{S}(1).$$

See the next section for precise meaning of the notations. Then we use the explicit formula for the action of τ on a standard function (eq. 2.4.1) to find the possible support points of the image of an element of C_k in the tree of SL_2. This yields the inclusions

$$\tau(\mathrm{C}_0) \subset \mathrm{C}_1 \qquad \text{and} \qquad \tau(\mathrm{C}_k) \subset \mathrm{C}_{k-1} \oplus \mathrm{C}_k \oplus \mathrm{C}_{k+1} \quad \text{for } k \geq 1.$$

The injectivity of $\tau : \mathrm{C}_0 \to \mathrm{C}_1$ is proved in the technical Lemma 4.5. Finally, we compute the explicit action of τ on the pro-p-Iwahori invariants and use this computation to establish the *filtration compatibility* condition of the abstract setup. Thus we have the desired freeness.

2. Preliminaries

2.1. **General notions.** – We take p to be a prime throughout, and $\bar{\mathbf{F}}_p$ a fixed algebraic closure of the finite field $\mathbf{F}_p$ with p elements. All representations, unless otherwise mentioned, are considered over $\bar{\mathbf{F}}_p$. We recall some generalities on the abstract representation theory of locally profinite groups. We let G be any locally profinite group, and H some closed subgroup. A representation π of G is called *smooth* if every vector $v \in \pi$ is fixed by some compact open subgroup of G. Let σ be a smooth representation of H. We consider the following space of functions :

$$\mathrm{IND}_\mathrm{H}^\mathrm{G}(\sigma) := \{f : \mathrm{G} \to \sigma \,|\, f(hg) = \sigma(h)(f(g)),\ \forall g \in \mathrm{G}, h \in \mathrm{H}\}.$$

Then, G acts on $\mathrm{IND}_\mathrm{H}^\mathrm{G}(\sigma)$ via $(g \cdot f)(g') := f(g'g)$. The *smooth part* of $\mathrm{IND}_\mathrm{H}^\mathrm{G}(\sigma)$, that is, vectors that have open stabilizers, is denoted by $\mathrm{Ind}_\mathrm{H}^\mathrm{G}(\sigma)$, and this subrepresentation is called the *smooth induction* of σ. The subrepresentation of $\mathrm{Ind}_\mathrm{H}^\mathrm{G}(\sigma)$ consisting of functions f such that the image of its support $\mathrm{Supp}(f)$ inside $\mathrm{H}\backslash\mathrm{G}$ is compact (equivalently, finite, whenever H is also open) is denoted by $\text{c-}\mathrm{Ind}_\mathrm{H}^\mathrm{G}(\sigma)$ or $\mathrm{ind}_\mathrm{H}^\mathrm{G}(\sigma)$, and is called the *compact induction* of σ.

In practice, whenever we use compact induction the subgroup H is typically considered to be open as well. So, for the remaining part of this subsection we take H to be an open subgroup of G. Then, by virtue of the H-linearity, the support of any $f \in \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$ can be written as a finite disjoint union of right H-cosets. We define some standard functions in $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$. For $g \in \mathrm{G}$ and $v \in \sigma$ we define :

$$[g, v](x) := \begin{cases} \sigma(xg)(v) & \text{if } x \in \mathrm{H}g^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

It can be checked that $g \cdot [g', v] = [gg', v]$ and $[gh, v] = [g, \sigma(h)(v)]$ for every $g, g' \in \mathrm{G}$ and $h \in \mathrm{H}$. Also, any $f \in \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$ can be written as

$$f = \sum_{\mathrm{H}g \in \mathrm{Supp}(f)} [g^{-1}, f(g)].$$

2.2. **Some standard notations related to** $\mathrm{SL}_2(\mathrm{F})$**.** – Let F be a non-archimedean local field of residue characteristic p. We denote its valuation ring by $\mathcal{O}_{\mathrm{F}}$ and its valuation ideal by $\mathfrak{p}_{\mathrm{F}}$ and we fix a *uniformizer* ϖ_{F} that generates $\mathfrak{p}_{\mathrm{F}}$. The residue field $k_{\mathrm{F}} := \mathcal{O}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}$ is a finite field of cardinality q which is a power of p. We set $\mathrm{G_S} := \mathrm{SL}_2(\mathrm{F})$ throughout this article. It is known that $\mathrm{G_S}$ has two maximal compact open subgroups $\mathrm{K}_0 := \mathrm{SL}_2(\mathcal{O}_{\mathrm{F}})$ and $\mathrm{K}_1 = \alpha \mathrm{K}_0 \alpha^{-1}$ where $\alpha := \mathrm{diag}(1, \varpi_{\mathrm{F}}) \in \mathrm{GL}_2(\mathrm{F})$. Let $\mathrm{I_S}(1)$ denote the *pro-p-Iwahori subgroup* of $\mathrm{G_S}$. We have

$$\mathrm{I_S}(1) := \begin{pmatrix} 1 + \mathfrak{p}_{\mathrm{F}} & \mathcal{O}_{\mathrm{F}} \\ \mathfrak{p}_{\mathrm{F}} & 1 + \mathfrak{p}_{\mathrm{F}} \end{pmatrix} \cap \mathrm{K}_0.$$

We finally let $\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^n)$ (resp. $\bar{\mathrm{U}}(\mathfrak{p}_{\mathrm{F}}^n)$) denote the upper triangular (resp. lower triangular) matrices in $\mathrm{G_S}$ with top right (resp. bottom left) entry in the fractional ideal $\mathfrak{p}_{\mathrm{F}}^n$ for $n \in \mathbf{Z}$.

2.3. **Generators of the weights of** $\mathrm{SL}_2(\mathcal{O}_{\mathrm{F}})$**.** – Let $k_{\mathrm{F}} = \mathbf{F}_q$ and $q = p^n$. Then it is well known that the irreducible $\bar{\mathbf{F}}_p$-representations of $\mathrm{SL}_2(\mathbf{F}_q)$ (or equivalently, smooth irreducible $\bar{\mathbf{F}}_p$-representations of $\mathrm{SL}_2(\mathcal{O}_{\mathrm{F}})$) are precisely of the form

$$\mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2 := \mathrm{Sym}^{r_0} \bar{\mathbf{F}}_p^2 \otimes \cdots \otimes \mathrm{Sym}^{r_{n-1}} \bar{\mathbf{F}}_p^2,$$

where $\vec{r} = (r_0, \ldots, r_{n-1}) \in \{0, \ldots, p-1\}^n$ and $\mathrm{SL}_2(\mathbf{F}_q)$ acts on $\mathrm{Sym}^{r_i} \bar{\mathbf{F}}_p^2 := \bigoplus_{l=0}^{r_i} \bar{\mathbf{F}}_p \mathrm{X}^{r_i - l} \mathrm{Y}^l$ via :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\mathrm{X}^{r_i - l} \mathrm{Y}^l) := (a^{p^i} \mathrm{X} + c^{p^i} \mathrm{Y})^{r_i - l} (b^{p^i} \mathrm{X} + c^{p^i} \mathrm{Y})^l.$$

Consider the element $\mathrm{X}^{\vec{r}} := \mathrm{X}^{r_0} \otimes \mathrm{X}^{r_1} \otimes \cdots \otimes \mathrm{X}^{r_{n-1}} \in \mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2$. The following result is well known. We will however give a proof to keep things self-contained, and also because some arguments in the proof will be used later.

Proposition 2.1. — *The representation* $\mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2$ *is generated by* $\mathrm{X}^{\vec{r}}$ *as an* $\bar{\mathbf{F}}_p[\bar{\mathrm{U}}]$*-module, and the line* $\bar{\mathbf{F}}_p \cdot \mathrm{X}^{\vec{r}}$ *is the unique* U*-invariant line in* $\mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2$*. Here,* U *(resp.* $\bar{\mathrm{U}}$*) denotes the upper (resp. lower) unipotent subgroup of* $\mathrm{SL}_2(\mathbf{F}_q)$*.*

Consequently, the set $\{\bar{u} \cdot \mathrm{X}^{\vec{r}} \mid \bar{u} \in \bar{\mathrm{U}}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})\}$ *spans* $\mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2$*, and the line* $\bar{\mathbf{F}}_p \cdot \mathrm{X}^{\vec{r}}$ *is the unique* $\mathrm{I_S}(1)$*-invariant line in* $\sigma_{\vec{r}}$*.*

Proof. Write $r = r_0 + r_1 p + \cdots + r_{n-1} p^{n-1}$; then $0 \le r < q = p^n$. Consider $\mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2$ as an $\mathrm{SL}_2(\mathbf{F}_q)$-subrepresentation of $\mathrm{Sym}^r \bar{\mathbf{F}}_p^2$ via the map $v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1} \mapsto v_0 v_1^p \cdots v_{n-1}^{p^{n-1}}$. Then, a basis of $\mathrm{Sym}^{\vec{r}} \bar{\mathbf{F}}_p^2$ is given by the family of monomials $\mathrm{X}^{\sum_{j=0}^{n-1} i_j p^j} \mathrm{Y}^{r - (\sum_{j=0}^{n-1} i_j p^j)}$, with $0 \le i_j \le r_j$ for all $j \in \{0, \ldots, n-1\}$.

Now, we recall that for $i = i_0 + i_1 p + \cdots + i_{n-1}p^{n-1} \leq r = r_0 + r_1 p + \cdots + r_{n-1}p^{n-1}$, the binomial coefficient $\binom{r}{i}$ in $\bar{\mathbf{F}}_p$ is given by

$$\binom{r}{i} = \begin{cases} \binom{r_0}{i_0}\binom{r_1}{i_1}\cdots\binom{r_{n-1}}{i_{n-1}} & \text{if } i_l \leq r_l \text{ for all } l \\ 0 & \text{otherwise} \end{cases}.$$

This simply follows by writing $(\mathrm{X}+1)^r \in \bar{\mathbf{F}}_p[\mathrm{X}]$ as $(\mathrm{X}+1)^{r_0}(\mathrm{X}^p+1)^{r_1}\cdots(\mathrm{X}^{p^{n-1}}+1)^{r_{n-1}}$ and then expanding and comparing the binomial coefficients.

Consequently, the vector $\mathrm{X}^i\mathrm{Y}^{r-i}$ (for $0 \leq i \leq r$) is in the above basis if and only if $\binom{r}{i} \neq 0$. Therefore, for any $\lambda \in \mathbf{F}_q$ we take $\bar{u}(\lambda) \in \bar{\mathrm{U}}$, and then we have

$$\bar{u}(\lambda)\cdot \mathrm{X}^r = \sum_{i=0}^{r}\binom{r}{i}\lambda^{r-i}\mathrm{X}^i\mathrm{Y}^{r-i} \in \bar{\mathbf{F}}_p[\bar{\mathrm{U}}]\cdot \mathrm{X}^r,$$

and the non-zero terms of the above sum correspond to those i for which $\binom{r}{i} \neq 0$. Since λ has q many possible values and $r < q$, an elementary argument in linear algebra shows that for each $i \leq r$ with $\binom{r}{i} \neq 0$, we have $\mathrm{X}^i\mathrm{Y}^{r-i} \in \bar{\mathbf{F}}_p[\bar{\mathrm{U}}]\cdot \mathrm{X}^r$, and hence $\mathrm{Sym}^{\vec{r}}\,\bar{\mathbf{F}}_p^2 = \bar{\mathbf{F}}_p[\bar{\mathrm{U}}]\cdot \mathrm{X}^{\vec{r}}$.

We now turn to the second statement to show that $\bar{\mathbf{F}}_p\cdot \mathrm{X}^{\vec{r}}$ is the unique U-invariant line in $\mathrm{Sym}^{\vec{r}}\bar{\mathbf{F}}_p^2$. As before, we think of $\mathrm{Sym}^{\vec{r}}\bar{\mathbf{F}}_p^2$ inside $\mathrm{Sym}^r\bar{\mathbf{F}}_p^2$, and consider a vector $v \in \mathrm{Sym}^{\vec{r}}\bar{\mathbf{F}}_p^2 \setminus \{\bar{\mathbf{F}}_p\cdot \mathrm{X}^r\}$. We can write $v = \sum_{\vec{i}\preceq\vec{r}}\mu_{\vec{i}}\mathrm{X}^{\sum_{j=0}^{n-1} i_j p^j}\mathrm{Y}^{r-(\sum_{j=0}^{n-1} i_j p^j)}$, where for the tuples $\vec{i} = (i_0,\ldots,i_{n-1})$ and $\vec{r} = (r_0,\ldots,r_{n-1})$ the notation $\vec{i}\preceq\vec{r}$ means $i_j \leq r_j$ for every $j \in \{0,\ldots,n-1\}$, and $\mu_{\vec{i}}$ are scalars such that $\mu_{\vec{i}} \neq 0$ for some $\vec{i}$. Then, for any $\lambda \in \mathbf{F}_q$, we have for $u(\lambda) \in \mathrm{U}$ the following

$$\begin{aligned} u(\lambda)\cdot v &= \sum_{i\preceq\vec{r}}\mu_{\vec{i}}\mathrm{X}^{\sum_{j=0}^{n-1} i_j p^j}(\lambda\mathrm{X}+\mathrm{Y})^{r-(\sum_{j=0}^{n-1} i_j p^j)} \\ &= \sum_{\vec{i}\preceq\vec{r}}\sum_{\vec{i}+\vec{j}\preceq\vec{r}}\mu_{\vec{i}}\lambda^j\binom{r-i}{j}\mathrm{X}^{i+j}\mathrm{Y}^{r-(i+j)} \\ &= \sum_{\vec{i}\preceq\vec{r}}\lambda^i\underbrace{\sum_{\vec{i}+\vec{j}\preceq\vec{r}}\mu_{\vec{j}}\binom{r-j}{i}\mathrm{X}^{i+j}\mathrm{Y}^{r-(i+j)}}_{:=v_{\vec{i}}}, \end{aligned}$$

where in the second equality the integers i and j are the p-ary sums corresponding to the tuples $\vec{i}$ and $\vec{j}$ respectively, and in the third equality we have interchanged the tuples $\vec{i}$ and $\vec{j}$. Now, since $v \notin \bar{\mathbf{F}}_p\cdot \mathrm{X}^r$, there is some tuple $\vec{j_0} \npreceq \vec{r}$ such that $\mu_{\vec{j_0}} \neq 0$. We take such a $\vec{j_0}$ with the corresponding p-ary sum j_0 being least. Consider two distinct tuples $\vec{i_1}$, $\vec{i_2} \preceq \vec{r}-\vec{j_0}$ so that the polynomials $v_{\vec{i_1}}$ and $v_{\vec{i_2}}$ are non-zero and hence linearly independent (by comparing the monomials of least degree in X). As a result, the subspace $\bar{\mathbf{F}}_p[\mathrm{U}]\cdot v$ has dimension at least two. ☐

Notation 2.2. – We write $\sigma_{\vec{r}}$ for the representation $\mathrm{Sym}^{\vec{r}}\bar{\mathbf{F}}_p^2$, and $v_{\sigma_{\vec{r}}}$ for the generating vector $\mathrm{X}^{\vec{r}}$ throughout this article. When $0 \leq r < p$ we will write r instead of $\vec{r}$.

Remark 2.3. – Similarly one can show that $\mathrm{Y}^{\vec{r}}$ also generates $\sigma_{\vec{r}}$ as an $\bar{\mathbf{F}}_p[\mathrm{U}]$-module, and the line $\bar{\mathbf{F}}_p\cdot \mathrm{Y}^{\vec{r}}$ is the unique $\bar{\mathrm{U}}$-invariant (equivalently $\mathrm{I}_{\mathrm{S}}(1)^{\mathsf{T}}$-invariant) line in $\sigma_{\vec{r}}$. Hence, the set $\{u\cdot(w_0\cdot v_{\sigma_{\vec{r}}})\,|\,u \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})\}$ spans $\sigma_{\vec{r}}$.

2.4. **Spherical Hecke algebra.** – We recall that for a weight $\sigma_{\vec{r}}$ of $\mathrm{K}_0 := \mathrm{SL}_2(\mathcal{O}_{\mathrm{F}})$, the *spherical Hecke algebra* $\mathcal{H}(\mathrm{G}_{\mathrm{S}},\mathrm{K}_0,\sigma_{\vec{r}}) := \mathrm{End}_{\mathrm{G}_{\mathrm{S}}}(\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G}_{\mathrm{S}}}(\sigma_{\vec{r}}))$ is generated by a single operator τ as a polynomial algebra in one variable i.e. $\mathrm{End}_{\mathrm{G}_{\mathrm{S}}}(\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G}_{\mathrm{S}}}(\sigma_{\vec{r}})) = \bar{\mathbf{F}}_p[\tau]$. In fact, we can compute the explicit action of τ on a standard function $[g,v] \in \mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G}_{\mathrm{S}}}(\sigma_{\vec{r}})$, as follows (see [1, Corollaire 3.12]) :

(2.4.1)

$$\boxed{\tau([g,v]) = \sum_{\lambda\in k_{\mathrm{F}}^2}\left[g\begin{pmatrix}1 & \mathrm{A}(\lambda)\\ 0 & 1\end{pmatrix}\alpha_0^{-1}, w_0\mathrm{U}_{\vec{r}}\sigma_{\vec{r}}\Big(\begin{pmatrix}0 & 1\\ -1 & \mathrm{A}(\lambda)\end{pmatrix}\Big)v\right] + \sum_{\mu\in k_{\mathrm{F}}}\left[g\begin{pmatrix}1 & 0\\ \varpi_{\mathrm{F}}\mathrm{A}(\mu) & 1\end{pmatrix}\alpha_0, \mathrm{U}_{\vec{r}}v\right].}$$

Here, for $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}) \in k_{\mathrm{F}}^m$ the notation $\mathrm{A}(\lambda) := \sum_{j=0}^{m-1}[\lambda_j]\varpi_{\mathrm{F}}^j$, and $\alpha_0 := \mathrm{diag}(\varpi_{\mathrm{F}}^{-1}, \varpi_{\mathrm{F}})$. The operator $\mathrm{U}_{\vec{r}} := \mathrm{U}_{r_0}\otimes\cdots\otimes\mathrm{U}_{r_{n-1}} \in \mathrm{End}_{\bar{\mathbf{F}}_p}(\sigma_{\vec{r}})$ is such that $\mathrm{U}_{r_j} \in \mathrm{End}_{\bar{\mathbf{F}}_p}(\sigma_{r_j})$ and is defined as follows :

$$\mathrm{U}_{r_j}(\mathrm{X}^l\mathrm{Y}^{r_j-l}) = \begin{cases}\mathrm{Y}^{r_j} & \text{if } l = 0\\ 0 & \text{if } l \neq 0\end{cases}. \tag{2.4.2}$$

Now the compact induction $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$ is a left $\mathcal{H}(\mathrm{G_S}, \mathrm{K}_0, \sigma_{\vec{r}})$-module. The mod p irreducibles of K_1 are denoted by $\sigma_{\vec{r}}^{\alpha}$. These have $\sigma_{\vec{r}}$ as the underlying representation space on which an element $k_1 = \alpha k_0\alpha^{-1} \in \mathrm{K}_1$ (where $k_0 \in \mathrm{K}_0$) acts by k_0. Since canonically we have $\mathcal{H}(\mathrm{G_S}, \mathrm{K}_0, \sigma_{\vec{r}}) \simeq \mathcal{H}(\mathrm{G_S}, \mathrm{K}_1, \sigma_{\vec{r}}^{\alpha})$ and $\mathrm{ind}_{\mathrm{K}_1}^{\mathrm{G_S}}(\sigma_{\vec{r}}^{\alpha}) \simeq \mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})^{\alpha}$ (see [1, Propositions 3.6 and 3.23]) it suffices to consider only the representation $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$ as a left $\mathcal{H}(\mathrm{G_S}, \mathrm{K}_0, \sigma_{\vec{r}})$-module. The $\mathcal{H}(\mathrm{G_S}, \mathrm{K}_1, \sigma_{\vec{r}}^{\alpha})$-module structure of $\mathrm{ind}_{\mathrm{K}_1}^{\mathrm{G_S}}(\sigma_{\vec{r}}^{\alpha})$ is essentially same.

2.5. **Pro-p-Iwahori invariants of $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$.** – We now recall some facts about the invariants $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})^{\mathrm{I_S}(1)}$. It is known that it has a basis consisting of functions $\{f_n \,|\, n \in \mathbf{Z}\}$ such that the function f_n is supported on $\mathrm{K}_0\alpha_0^{-n}\mathrm{I_S}(1)$ and satisfies

$$f_n(\alpha_0^{-n}) = \begin{cases} w_0\cdot v_{\sigma_{\vec{r}}} & \text{if } n > 0\\ v_{\sigma_{\vec{r}}} & \text{if } n \le 0\end{cases}.$$

Now, as $\mathrm{I_S}(1) = \mathrm{U_S}(\mathcal{O}_{\mathrm{F}}) \times \mathrm{T_S}(1+\mathfrak{p}_{\mathrm{F}}) \times \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})$, and α_0 normalizes $\mathrm{T_S}$, we have $\mathrm{K}_0\alpha_0^{-n}\mathrm{I_S}(1) = \mathrm{K}_0\alpha_0^{-n}\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})$ and $\mathrm{K}_0\alpha_0^{n}\mathrm{I_S}(1) = \mathrm{K}_0\alpha_0^{n}\mathrm{U_S}(\mathcal{O}_{\mathrm{F}})$ for $n > 0$. Therefore, we have the decompositions

$$\mathrm{K}_0\alpha_0^{-n}\mathrm{I_S}(1) = \bigcup_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2n})} \mathrm{K}_0\alpha_0^{-n}\bar{u}$$

and

$$\mathrm{K}_0\alpha_0^{n}\mathrm{I_S}(1) = \bigcup_{u\in\mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2n})} \mathrm{K}_0\alpha_0^{n}u$$

for $n > 0$, and hence for $n \in \mathbf{Z}$ we can write these f_n as follows :

$$\boxed{f_n = \begin{cases}\sum_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2n})}[\bar{u}\alpha_0^n, w_0\cdot v_{\sigma_{\vec{r}}}] & \text{if } n > 0\\ \sum_{u\in\mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{-2n})}[u\alpha_0^n, v_{\sigma_{\vec{r}}}] & \text{if } n \le 0\end{cases}.} \tag{2.5.1}$$

3. An abstract linear-algebraic criterion

In this section we prove a purely linear-algebraic/combinatorial lemma that forms the abstract framework for the results of the subsequent section. This result gives certain conditions to test when a graded vector space with a finite collection of commuting linear operators is a free module over the algebra generated by the operators.

Lemma 3.1. — *Let $d \ge 1$ and V be a vector space over a field k and let*

$$\mathrm{V} = \bigoplus_{\mathbf{n}\in\mathbf{Z}_{\ge 0}^d} \mathrm{C}_{\mathbf{n}}$$

be a direct sum decomposition by non-trivial subspaces indexed by multi-indices $\mathbf{n} = (n_1, \ldots, n_d)$; set $|\mathbf{n}| := n_1 + \cdots + n_d$. Let $\mathrm{T}_1, \ldots, \mathrm{T}_d$ be linear operators on V which commute pairwise : $\mathrm{T}_i\mathrm{T}_j = \mathrm{T}_j\mathrm{T}_i$ for all i, j. Assume the following :

(H1) *Each* $\mathrm{C}_{\mathbf{n}}$ *is finite dimensional and* $\dim \mathrm{C}_{\mathbf{n}} > \sum_{\{\mathbf{m} \mid |\mathbf{m}|<|\mathbf{n}|\}} \dim \mathrm{C}_{\mathbf{m}}$.

(H2) *For each coordinate* j,

$$\mathrm{T}_j(\mathrm{C}_{\mathbf{0}}) \subseteq \mathrm{C}_{\mathbf{e}_j}, \qquad \textit{and} \qquad \mathrm{T}_j|_{\mathrm{C}_{\mathbf{0}}} : \mathrm{C}_{\mathbf{0}} \to \mathrm{C}_{\mathbf{e}_j} \textit{ is injective}$$

where $\mathbf{e}_j$ *denotes the* d*-tuple having* 1 *in the* j*-th coordinate and zero elsewhere.*

(H3) *For every multi-index* $\mathbf{n} \neq \mathbf{0}$ *and each* $j \in \{1, \ldots, d\}$,

$$\mathrm{T}_j(\mathrm{C}_{\mathbf{n}}) \subseteq \mathrm{C}_{\mathbf{n}-\mathbf{e}_j} \oplus \mathrm{C}_{\mathbf{n}} \oplus \mathrm{C}_{\mathbf{n}+\mathbf{e}_j},$$

where $\mathrm{C}_{\mathbf{n}-\mathbf{e}_j}$ *is taken to be* 0 *if* $n_j = 0$.

(H4) *For each* $\mathrm{N} \geq 0$, *write*

$$\mathrm{B}_{\mathrm{N}} := \bigoplus_{|\mathbf{n}| \leq \mathrm{N}} \mathrm{C}_{\mathbf{n}}.$$

Then, for each j *and each* $\mathrm{N} \geq 0$ *the following holds : if* $f \in \mathrm{B}_{\mathrm{N}+1}$ *and* $\mathrm{T}_j(f) \in \mathrm{B}_{\mathrm{N}+1}$, *then* $f \in \mathrm{B}_{\mathrm{N}}$.

(H5) *For each* $\mathbf{n}$ *with* $|\mathbf{n}| \geq 2$, *the collection of subspaces*

$$\{(\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}})(\mathrm{C}_{\mathbf{j}}) \mid \mathbf{j} < \mathbf{n} \quad \textit{coordinatewise}\ \} \qquad (\textit{where } \pi_{\mathbf{n}} : \mathrm{V} \to \mathrm{C}_{\mathbf{n}} \textit{ is the projection map})$$

are direct summands in $\mathrm{C}_{\mathbf{n}}$.

Then there exists non-empty subsets $\mathrm{A}_{\mathbf{n}} \subset \mathrm{C}_{\mathbf{n}}$ *for every* $\mathbf{n}$ *such that for any* $\mathrm{N} \geq 0$ *the collection*

$$\{\mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}}) \mid |\mathbf{i}| + |\mathbf{j}| \leq \mathrm{N}\} \qquad (\textit{where } \mathbf{T}^{\mathbf{i}} := \mathrm{T}_1^{i_1} \cdots \mathrm{T}_d^{i_d})$$

is a mutually disjoint collection of sets and their union forms a basis of B_{N}. *Consequently, if* V *is treated as a module over the polynomial algebra* $k[\mathrm{X}_1, \ldots, \mathrm{X}_d]$ *where* X_j *acts by* T_j, *then the union* $\bigsqcup_{\mathbf{n}} \mathrm{A}_{\mathbf{n}}$ *is a module basis of* V.

Proof. We will construct the sets $\mathrm{A}_{\mathbf{n}}$ inductively. Set $\mathrm{A}_{\mathbf{0}} \subset \mathrm{C}_{\mathbf{0}}$ to be a basis. Since $\dim \mathrm{C}_{\mathbf{e}_j} > \dim \mathrm{C}_{\mathbf{0}}$ for each j we can take $\mathrm{A}_{\mathbf{e}_j} \subset \mathrm{C}_{\mathbf{e}_j}$ to be a non-empty set extending the linearly independent set $\mathrm{T}_j(\mathrm{A}_{\mathbf{0}})$ to a basis of $\mathrm{C}_{\mathbf{e}_j}$. Thus we have constructed $\mathrm{A}_{\mathbf{n}}$ for all $\mathbf{n}$ with $|\mathbf{n}| = 1$.

Now, we define $\mathrm{top}(f) := \max\{|\mathbf{j}| \mid \pi_{\mathbf{j}}(f) \neq 0\}$. Then for each j and each $\mathrm{N} \geq 0$ we have the following condition

$$(\mathrm{H4}') \text{ — } \mathrm{top}(f) = \mathrm{N} \implies \mathrm{top}(\mathrm{T}_j(f)) = \mathrm{N} + 1.$$

For $\mathrm{N} = 0$ this is essentially the hypothesis (H2). For $\mathrm{N} \geq 1$ if $\mathrm{top}(f) = \mathrm{N}$ and $\mathrm{top}(\mathrm{T}_j(f)) \leq \mathrm{N}$ (note that by the hypothesis (H3) we know that $\mathrm{top}(\mathrm{T}_j(f)) \leq \mathrm{N} + 1$) then by the hypothesis (H4) we know $f \in \mathrm{B}_{\mathrm{N}-1}$ and hence $\mathrm{top}(f) \leq \mathrm{N} - 1$, a contradiction. Thus each T_j is injective. Thus, if $f \in \mathrm{C}_{\mathbf{j}}$, then the *top multi-index component* of $\mathbf{T}^{\mathbf{i}}(f)$ is *precisely* $\mathrm{C}_{\mathbf{i}+\mathbf{j}}$. Consequently, the maps $\pi_{\mathbf{i}+\mathbf{j}} \circ \mathbf{T}^{\mathbf{i}} : \mathrm{C}_{\mathbf{j}} \to \mathrm{C}_{\mathbf{i}+\mathbf{j}}$ are injective.

Now suppose $\mathrm{N} \geq 1$ and we have constructed non-empty subsets $\mathrm{A}_{\mathbf{j}} \subset \mathrm{C}_{\mathbf{j}}$ for all $\mathbf{j}$ with $|\mathbf{j}| \leq \mathrm{N}$ such that

$$\bigsqcup_{|\mathbf{i}|+|\mathbf{j}| \leq \ell} \mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}})$$

is a basis for B_{ℓ} for all $\ell \leq \mathrm{N}$. We consider the following set

$$\mathcal{E} := \bigsqcup_{\substack{|\mathbf{i}|+|\mathbf{j}|=\mathrm{N}+1 \\ |\mathbf{j}| \leq \mathrm{N}}} \mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}}).$$

At first, we observe that the sets $\mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}})$ appearing in the above union are indeed disjoint. Suppose $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{i}', \mathbf{j}')$ with $|\mathbf{i}| + |\mathbf{j}| = |\mathbf{i}'| + |\mathbf{j}'| = \mathrm{N} + 1$. We first consider the case when $\mathbf{j} = \mathbf{j}'$ so that $|\mathbf{i}| = |\mathbf{i}'|$ and $\mathbf{i} \neq \mathbf{i}'$. Then for $a, b \in \mathrm{A}_{\mathbf{j}}$ we look at the top multi-index of $\mathbf{T}^{\mathbf{i}}(a)$ which is $\mathbf{i} + \mathbf{j}$; whereas the top multi-index of the element $\mathbf{T}^{\mathbf{i}'}(b)$ is $\mathbf{i}' + \mathbf{j}$ and these are distinct. Next, consider the case when $\mathbf{j} \neq \mathbf{j}'$. In this case if $\mathbf{i} + \mathbf{j} \neq \mathbf{i}' + \mathbf{j}'$ then for $a \in \mathrm{A}_{\mathbf{j}}$ and $b \in \mathrm{A}_{\mathbf{j}'}$ the top components of $\mathbf{T}^{\mathbf{i}}(a)$ and $\mathbf{T}^{\mathbf{i}'}(b)$ are in $\mathrm{C}_{\mathbf{i}+\mathbf{j}}$ and $\mathrm{C}_{\mathbf{i}'+\mathbf{j}'}$ respectively, which are different. Finally, we consider the case when $\mathbf{j} \neq \mathbf{j}'$ and $\mathbf{n} := \mathbf{i} + \mathbf{j} = \mathbf{i}' + \mathbf{j}'$. Then $\mathbf{j}, \mathbf{j}' < \mathbf{n}$ coordinatewise. Hence, by the hypothesis (H5) the top components of $\mathbf{T}^{\mathbf{i}}(a)$ and $\mathbf{T}^{\mathbf{i}'}(b)$ are in different direct summands and hence they can never be equal.

Next, we show that $\mathcal{E}$ is linearly independent. Assume a linear relation of the form

$$\sum_{\{(\mathbf{i},\mathbf{j}) \,|\, |\mathbf{i}|+|\mathbf{j}|=\mathrm{N}+1,\, |\mathbf{j}|\leq \mathrm{N}\}} \sum_{a \in \mathrm{A}_{\mathbf{j}}} \lambda_{\mathbf{i},\mathbf{j},a} \mathbf{T}^{\mathbf{i}}(a) = 0.$$

Fix any multi-index $\mathbf{n}$ with $|\mathbf{n}| = \mathrm{N}+1$. We project onto $\mathrm{C}_{\mathbf{n}}$ and obtain

$$\sum_{\{\mathbf{j}<\mathbf{n} \,|\, |\mathbf{j}|\leq \mathrm{N}\}} (\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}}) \left(\sum_{a \in \mathrm{A}_{\mathbf{j}}} \lambda_{\mathbf{n}-\mathbf{j},\mathbf{j},a} a \right) = 0.$$

Now since for each $\mathbf{j} \in \{\mathbf{j} < \mathbf{n} \,|\, |\mathbf{j}| \leq \mathrm{N}\}$ we know $(\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}}) \left(\sum_{a\in \mathrm{A}_{\mathbf{j}}} \lambda_{\mathbf{n}-\mathbf{j},\mathbf{j},a} a\right) \in (\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}})(\mathrm{C}_{\mathbf{j}})$ and these are direct summands, therefore by injectivity of $\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}} : \mathrm{C}_{\mathbf{j}} \to \mathrm{C}_{\mathbf{n}}$ we have $\sum_{a\in \mathrm{A}_{\mathbf{j}}} \lambda_{\mathbf{n}-\mathbf{j},\mathbf{j},a} a = 0$ for all such $\mathbf{j}$. Since by induction hypothesis $\mathrm{A}_{\mathbf{j}}$ are linearly independent we have each $\lambda_{\mathbf{n}-\mathbf{j},\mathbf{j},a} = 0$. Thus we have established linear independence of $\mathcal{E}$.

Now consider the family

$$\bigsqcup_{\{(\mathbf{i},\mathbf{j}) \,|\, |\mathbf{i}|+|\mathbf{j}|\leq \mathrm{N}\}} \mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}}) \sqcup \bigsqcup_{\{(\mathbf{i},\mathbf{j}) \,|\, |\mathbf{j}|\leq \mathrm{N},\, |\mathbf{i}|+|\mathbf{j}|=\mathrm{N}+1\}} \mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}}).$$

This set is linearly independent since the second set is linearly independent and has top components of degree $\mathrm{N}+1$. Now we fix $\mathbf{n}$ with $|\mathbf{n}| = \mathrm{N}+1$. We take $\mathrm{A}_{\mathbf{n}} \subset \mathrm{C}_{\mathbf{n}}$ to be a basis of a complement of

$$\mathrm{span}\left(\bigsqcup_{\{\mathbf{j}<\mathbf{n} \,|\, |\mathbf{j}|\leq \mathrm{N}\}} \left(\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}}\right)(\mathrm{A}_{\mathbf{j}}) \right).$$

Then the *growth* condition (H1) together with injectivity of the maps $\pi_{\mathbf{n}} \circ \mathbf{T}^{\mathbf{n}-\mathbf{j}}$ guarantee that $\mathrm{A}_{\mathbf{n}} \neq \emptyset$. Thus we obtain

$$\bigsqcup_{\{(\mathbf{i},\mathbf{j}) \,|\, |\mathbf{i}|+|\mathbf{j}|\leq \mathrm{N}\}} \mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}}) \sqcup \bigsqcup_{\{(\mathbf{i},\mathbf{j}) \,|\, |\mathbf{j}|\leq \mathrm{N},\, |\mathbf{i}|+|\mathbf{j}|=\mathrm{N}+1\}} \mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}}) \sqcup \bigsqcup_{\{\mathbf{n} \,|\, |\mathbf{n}|=\mathrm{N}+1\}} \mathrm{A}_{\mathbf{n}}$$

as a basis of $\mathrm{B}_{\mathrm{N}+1}$. ☐

Remark 3.2. – For $d = 1$ the hypothesis (H5) is not required. Note that it was invoked in the proof in two instances :

(1) In showing that

$$\{\mathrm{T}^{i}(\mathrm{A}_{j}) \,|\, i + j = \mathrm{N}+1,\, j \leq \mathrm{N}\}$$

for $\mathrm{N} \geq 1$ forms a disjoint collection of sets. This can be proved without (H5) as follows. Indeed, if $(i, j) \neq (i', j')$ then $j \neq j'$. For $a \in \mathrm{A}_j$ and $b \in \mathrm{A}_{j'}$ the equality

$$\mathrm{T}^{i}(a) = \mathrm{T}^{i'}(b)$$

gives a contradiction by combining injectivity of T and the fact that $\mathrm{T}^{i-1}(a), \mathrm{T}^{i'-1}(b) \in \bigsqcup_{i+j\leq \mathrm{N}} \mathrm{T}^{i}(\mathrm{A}_j)$; the disjointness of the family $\{\mathrm{T}^{i}(\mathrm{A}_j) \,|\, i + j \leq \mathrm{N}\}$ is assumed in the induction hypothesis.

(2) In showing that

$$\bigsqcup_{\{(i,j) \,|\, i+j\leq \mathrm{N}\}} \mathrm{T}^{i}(\mathrm{A}_j) \sqcup \bigsqcup_{\{(i,j) \,|\, i+j=\mathrm{N}+1,\, j\leq \mathrm{N}\}} \mathrm{T}^{i}(\mathrm{A}_j)$$

is linearly independent for $\mathrm{N} \geq 1$. Again this can be proved without (H5) by a combined use of (H4) and the induction hypothesis : that $\bigsqcup_{i+j\leq \ell} \mathrm{T}^{i}(\mathrm{A}_j)$ is a basis of B_ℓ for every $\ell \leq \mathrm{N}$. Indeed, by way of contradiction suppose we have a non-trivial linear combination of elements of the above set written as $f_1 + f_2 = 0$ where f_1 is a linear combination of elements from the first set and f_2 is a linear combination of elements from the second set. We can write f_2 as $\mathrm{T}(f_2')$ where f_2' is a linear combination of elements from $\bigsqcup_{j+i=\mathrm{N}} \mathrm{T}^i(\mathrm{A}_j)$ with same coefficients that appear in the combination representing f_2. Thus $f_2' \in \mathrm{B_N}$ and $\mathrm{T}(f_2') = -f_1 \in \mathrm{B_N}$ and by condition (3) we have $f_2' \in \mathrm{B_{N-1}}$ since we have assumed $\mathrm{N} \geq 1$. But $\bigsqcup_{j+i\leq \mathrm{N}} \mathrm{T}^i(\mathrm{A}_j) = \bigsqcup_{j+i\leq \mathrm{N}-1} \mathrm{T}^i(\mathrm{A}_j) \sqcup \bigsqcup_{j+i=\mathrm{N}} \mathrm{T}^i(\mathrm{A}_j)$ is a basis of $\mathrm{B_N}$ and $\bigsqcup_{j+i\leq \mathrm{N}-1} \mathrm{T}^i(\mathrm{A}_j)$ is a basis of $\mathrm{B_{N-1}}$ by the induction hypothesis. Therefore, all coefficients in the linear combination representing f_2' (and hence f_2) are zero. Consequently, all coefficients representing f_1 are also zero.

Hence, we can choose $\mathrm{A_{N+1}}$ to be a basis of a complement of

$$\mathrm{span}\left(\pi_{\mathrm{N}+1}\left(\bigsqcup_{\{(i,j)\,|\,i+j=\mathrm{N}+1,\, j\leq \mathrm{N}\}} \mathrm{T}^i(\mathrm{A}_j)\right)\right)$$

noting that $\mathrm{A_{N+1}} \neq \emptyset$, since

$$\pi_{\mathrm{N}+1}\left(\bigsqcup_{\{(i,j)\,|\,i+j=\mathrm{N}+1,\, j\leq \mathrm{N}\}} \mathrm{T}^i(\mathrm{A}_j)\right) = \pi_{\mathrm{N}+1}\left(\mathrm{T}\left(\bigsqcup_{\{(i,j)\,|\,i+j=\mathrm{N},\, j\leq \mathrm{N}\}} \mathrm{T}^i(\mathrm{A}_j)\right)\right) \subset (\pi_{\mathrm{N}+1}\circ \mathrm{T})(\mathrm{C_N}) \subsetneq \mathrm{C_{N+1}}$$

because $\pi_{\mathrm{N}+1}\circ \mathrm{T}$ is injective and $\dim \mathrm{C_N} < \dim \mathrm{C_{N+1}}$.

However, when $d > 1$ then without (H5) we can not ascertain the first point made above. On a thorough scrutiny of the proof it will become clear that (H5) is used really to overcome the problem of having to show that : for two pairs $(\mathbf{i},\mathbf{j})$ and $(\mathbf{i}',\mathbf{j}')$ (with $|\mathbf{j}|, |\mathbf{j}'| \leq \mathrm{N}$, $|\mathbf{i}|+|\mathbf{j}| = |\mathbf{i}'|+|\mathbf{j}'| = \mathrm{N}+1$) satisfying $\mathbf{i}+\mathbf{j} = \mathbf{i}'+\mathbf{j}'$ and $\mathbf{j}\neq \mathbf{j}'$, the sets $\mathbf{T}^{\mathbf{i}}(\mathrm{A}_{\mathbf{j}})$ and $\mathbf{T}^{\mathbf{i}'}(\mathrm{A}_{\mathbf{j}'})$ are disjoint. More specifically, when $\mathbf{i}$ is of the form $(*,0,*,0,\dots)$ and $\mathbf{i}'$ is of the form $(0,*,0,*,\dots)$, that is there does not exist any coordinate l for which i_l, i_l' are both positive. Otherwise, if such a coordinate l existed, then we could have used injectivity of T_l and the induction hypothesis to conclude the disjointness. For example, if $d = 2$ and $a \in \mathrm{C}_{(0,1)}, b \in \mathrm{C}_{(1,0)}$ then without (H5) there is no way to ensure that $\mathrm{T}_1(a)$ does not coincide with $\mathrm{T}_2(b)$ in $\mathrm{C}_{(1,1)}$ (note that $\mathrm{T}_1(a) \in \mathrm{C}_{(0,1)} \oplus \mathrm{C}_{(1,1)}$ and $\mathrm{T}_2(b) \in \mathrm{C}_{(1,0)} \oplus \mathrm{C}_{(1,1)}$, thus if they were to be equal then they must be elements of $\mathrm{C}_{(1,1)}$).

4. Freeness of $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$ as a Hecke module

In this section we take $\mathrm{V} = \mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$ and $\mathrm{T} = \tau$ in the framework of Lemma 3.1. Here obviously $d = 1$. At first we introduce some new objects and notations borrowed from [12, 14]. First recall that we have the well known Cartan-Iwahori decomposition : $\mathrm{G_S} = \bigsqcup_{n\in\mathbf{Z}} \mathrm{K}_0\alpha_0^n \mathrm{I_S}(1)$. For $n \geq 0$, we denote by $\mathrm{R}_n^+(\sigma_{\vec{r}})$ (resp. $\mathrm{R}_n^-(\sigma_{\vec{r}})$) the subspace of functions in $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G_S}}(\sigma_{\vec{r}})$ which are supported on $\mathrm{K}_0\alpha_0^n\mathrm{I_S}(1) = \mathrm{K}_0\alpha_0^n\mathrm{U_S}(\mathcal{O}_\mathrm{F})$ (resp. $\mathrm{K}_0\alpha_0^{-(n+1)}\mathrm{I_S}(1) = \mathrm{K}_0\alpha_0^{-(n+1)}\bar{\mathrm{U}}_\mathrm{S}(\mathfrak{p}_\mathrm{F})$). We write $\mathrm{R}_n^+(\sigma_{\vec{r}}) = [\mathrm{U_S}(\mathcal{O}_\mathrm{F})\alpha_0^{-n}, \sigma_{\vec{r}}]$ for $n \geq 0$, and $\mathrm{R}_{n-1}^-(\sigma_{\vec{r}}) = [\bar{\mathrm{U}}_\mathrm{S}(\mathfrak{p}_\mathrm{F})\alpha_0^n, \sigma_{\vec{r}}]$ for $n \geq 1$. By convention $\mathrm{R}_{-1}^-(\sigma_{\vec{r}}) := \mathrm{R}_0^+(\sigma_{\vec{r}})$. We also set $\mathrm{C}_{0,\sigma_{\vec{r}}} := \mathrm{R}_0^+(\sigma_{\vec{r}})$, $\mathrm{C}_{n,\sigma_{\vec{r}}} := \mathrm{R}_n^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_{n-1}^-(\sigma_{\vec{r}})$ for $n \geq 1$ and $\mathrm{B}_{n,\sigma_{\vec{r}}} := \bigoplus_{k\leq n} \mathrm{C}_{k,\sigma_{\vec{r}}}$ for $n \geq 0$.

As mentioned in Remark 3.2, we need to check the following conditions are satisfied

- (C1) Each $\mathrm{C}_{n,\sigma_{\vec{r}}}$ is finite dimensional and $\dim \mathrm{C}_{n,\sigma_{\vec{r}}} > \sum_{\{k\,|\,k<n\}} \mathrm{C}_{k,\sigma_{\vec{r}}}$.
- (C2) $\tau(\mathrm{C}_{0,\sigma_{\vec{r}}}) \subset \mathrm{C}_{1,\sigma_{\vec{r}}}$ and $\tau|_{\mathrm{C}_{0,\sigma_{\vec{r}}}} : \mathrm{C}_{0,\sigma_{\vec{r}}} \to \mathrm{C}_{1,\sigma_{\vec{r}}}$ is injective.
- (C3) For every $n \geq 1$ we have $\tau(\mathrm{C}_{n\sigma_{\vec{r}}}) \subset \mathrm{C}_{n-1,\sigma_{\vec{r}}} \oplus \mathrm{C}_{n,\sigma_{\vec{r}}} \oplus \mathrm{C}_{n+1,\sigma_{\vec{r}}}$.
- (C4) For each $n \geq 0$: if $f \in \mathrm{B}_{n+1,\sigma_{\vec{r}}}$ and $\tau(f) \in \mathrm{B}_{n+1,\sigma_{\vec{r}}}$ then $f \in \mathrm{B}_{n,\sigma_{\vec{r}}}$.

See the following remark for (C1).

Remark 4.1. – It is often useful to interpret the subspaces $\mathrm{C}_{n,\sigma_{\vec{r}}}$ and $\mathrm{B}_{n,\sigma_{\vec{r}}}$ in terms of the Bruhat-Tits tree of $\mathrm{SL}_2(\mathrm{F})$. For a basic introduction to Bruhat-Tits theory with the example of SL_2 worked out in full detail one can see [13, Section 4.2]; also the recent work [9] works out the example SL_n with copious details. Note that $\mathrm{C}_{0,\sigma_{\vec{r}}}$ is the subspace of all functions supported on the central vertex corresponding to the subgroup K_0. For $n \geq 1$ the subspace $\mathrm{C}_{n,\sigma_{\vec{r}}}$ consists of all functions supported on vertices which are at a distance of $2n$ from the central vertex. Since each vertex of the tree has degree $q := |k_{\mathrm{F}}|$, we know that $\dim \mathrm{C}_{n,\sigma_{\vec{r}}} = (q+1)q^{2n-1} \cdot \dim \sigma_{\vec{r}}$ for $n \geq 1$ and $\dim \mathrm{C}_{0,\sigma_{\vec{r}}} = \dim \sigma_{\vec{r}}$. Hence the inequality

$$\dim \mathrm{C}_{n,\sigma_{\vec{r}}} > \sum_{m=0}^{n-1} \dim \mathrm{C}_{m,\sigma_{\vec{r}}}$$

is clear when $n = 1$. For $n \geq 2$ we have to show

$$(q+1)q^{2n-1} > 1 + (q+1) \cdot q \cdot \frac{q^{2(n-1)} - 1}{q^2 - 1},$$

which can be easily proved using elementary algebra noting that $q \geq 2$. Thus the condition (C1) above checks out.

Remark 4.2. – Also note that for $n \geq 0$ both the spaces $\mathrm{R}_n^+(\sigma_{\vec{r}})$ and $\mathrm{R}_{n-1}^-(\sigma_{\vec{r}})$ are $\mathrm{I}_{\mathrm{S}}(1)$-stable. For $n \geq 0$ we have $f_{-n} \in \mathrm{R}_n^+(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$, and for $n \geq 1$ we have $f_n \in \mathrm{R}_{n-1}^-(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$. Also, every function in $\mathrm{R}_n^+(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$ (resp. $\mathrm{R}_{n-1}^-(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$) is determined by its value on α_0^n (resp. α_0^{-n}). Hence, the spaces $\mathrm{R}_n^+(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$ and $\mathrm{R}_{n-1}^-(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$ are one dimensional and generated by f_{-n} and f_n respectively.

We now state the main theorem of this article.

Theorem 4.3. — *The compactly induced representation* $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G}_{\mathrm{S}}}(\sigma_{\vec{r}})$ *is a free module of infinite rank over the spherical Hecke algebra* $\mathcal{H}(\mathrm{G}_{\mathrm{S}}, \mathrm{K}_0, \sigma_{\vec{r}})$.

To prove this we will show that the Hecke operator τ satisfies the conditions (C2)-(C4) above.

4.1. **Action of τ on the subspaces $\mathrm{C}_{n,\sigma_{\vec{r}}}$.** We will now show that the Hecke operator τ satisfies the conditions (C2) and (C3).

Lemma 4.4. — *We have the following :*

(i) $\tau(\mathrm{R}_0^+(\sigma_{\vec{r}})) \subseteq \mathrm{R}_1^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_0^-(\sigma_{\vec{r}})$.

(ii) $\tau(\mathrm{R}_n^+(\sigma_{\vec{r}})) \subseteq \mathrm{R}_{n-1}^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_n^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_{n+1}^+(\sigma_{\vec{r}})$, *for* $n \geq 1$.

(iii) $\tau(\mathrm{R}_n^-(\sigma_{\vec{r}})) \subseteq \mathrm{R}_{n-1}^-(\sigma_{\vec{r}}) \oplus \mathrm{R}_n^-(\sigma_{\vec{r}}) \oplus \mathrm{R}_{n+1}^-(\sigma_{\vec{r}})$, *for* $n \geq 0$.

Thus $\tau(\mathrm{C}_{0,\sigma_{\vec{r}}}) \subseteq \mathrm{C}_{1,\sigma_{\vec{r}}}$ *and* $\tau(\mathrm{C}_{n,\sigma_{\vec{r}}}) \subseteq \mathrm{C}_{n-1,\sigma_{\vec{r}}} \oplus \mathrm{C}_{n,\sigma_{\vec{r}}} \oplus \mathrm{C}_{n+1,\sigma_{\vec{r}}}$ *for* $n \geq 1$,

Proof. (i) It suffices to compute the action of τ on a standard function $[u, v] \in \mathrm{R}_0^+(\sigma_{\vec{r}})$, where $u \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})$. Using the formula 2.4.1 the required containment follows from the simple observation that if $\bar{u} \in \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})$, then $u\bar{u} \in \mathrm{I}_{\mathrm{S}}(1)$, so that we can write $u\bar{u} = \bar{u}_1 t u_2$, where $\bar{u}_1 \in \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})$, $t \in \mathrm{T}_{\mathrm{S}}(1+\mathfrak{p}_{\mathrm{F}})$, and $u_2 \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})$. But then again we can write $u_2\alpha_0 = \alpha_0 u_2'$ with $u_2' \in \mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^2) \subset \mathrm{K}_0$. Hence, the sum $\sum_\mu [u\bar{u}(\varpi_{\mathrm{F}}\mathrm{A}(\mu))\alpha_0, \mathrm{U}_{\vec{r}}v] \in \mathrm{R}_0^-(\sigma_{\vec{r}})$.

(ii) Again, take a standard function $[u(x)\alpha_0^{-n}, v] \in \mathrm{R}_n^+(\sigma_{\vec{r}})$ where $u(x)$ denotes an upper unipotent matrix with the top right entry $x \in \mathcal{O}_{\mathrm{F}}$. Here, we look at the formula 2.4.1 and consider the elements of the form $u(x)\alpha_0^{-n}u(\mathrm{A}(\lambda))\alpha_0^{-1}$ and $u(x)\alpha_0^{-n}\bar{u}(\varpi_{\mathrm{F}}\mathrm{A}(\mu))\alpha_0$ to determine which $\mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})$-$\mathrm{K}_0$ double coset they belong to. At first, note that $u(x)\alpha_0^{-n}u(\mathrm{A}(\lambda))\alpha_0^{-1} = u(x + \varpi_{\mathrm{F}}^{2n}\mathrm{A}(\lambda))\alpha_0^{-(n+1)} \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})\alpha_0^{-(n+1)}$. Next, for $\mu \in k_{\mathrm{F}}$ we can easily see by using elementary row and column reduction that :

$$\alpha_0^{-n}\bar{u}(\varpi_{\mathrm{F}}\mathrm{A}(\mu))\alpha_0 = \begin{pmatrix} \varpi_{\mathrm{F}}^{n-1} & 0 \\ \varpi_{\mathrm{F}}^{-n}\mathrm{A}(\mu) & \varpi_{\mathrm{F}}^{-n+1} \end{pmatrix} \in \begin{cases} \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})\alpha_0^{-(n-1)}\mathrm{K}_0 & \text{if } \mu = 0 \\ \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})\alpha_0^{-n}\mathrm{K}_0 & \text{if } \mu \neq 0 \end{cases}.$$

As a result we have $\tau(\mathrm{R}_n^+(\sigma_{\vec{r}})) \subseteq \mathrm{R}_{n-1}^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_n^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_{n+1}^+(\sigma_{\vec{r}})$, for $n \geq 1$.

(iii) As before we take a standard function $[\bar{u}(y)\alpha_0^{n+1}, v] \in \mathrm{R}_n^-(\sigma_{\vec{r}})$ where $\bar{u}(y)$ denotes a lower unipotent matrix with the bottom left entry $y \in \mathfrak{p}_{\mathrm{F}}$. In this case also we look at the formula 2.4.1 and consider the elements

$\bar{u}(y)\alpha_0^{n+1}u(\mathrm{A}(\lambda))\alpha_0^{-1}$ and $\bar{u}(y)\alpha_0^{n+1}\bar{u}(\varpi_{\mathrm{F}}\mathrm{A}(\mu))\alpha_0$. Then we determine which $\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})$-$\mathrm{K}_0$ double coset they belong to. First, note that $\bar{u}(y)\alpha_0^{n+1}\bar{u}(\varpi_{\mathrm{F}}\mathrm{A}(\mu))\alpha_0 = \bar{u}(y+\varpi_{\mathrm{F}}^{2n+3}\mathrm{A}(\mu))\alpha_0^{n+2} \in \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})\alpha_0^{n+2}$. Next, for $\lambda \in k_{\mathrm{F}}^2$ we have by row and column reduction the following :

$$\alpha_0^{n+1}u(\mathrm{A}(\lambda))\alpha_0^{-1} = \begin{pmatrix} \varpi_{\mathrm{F}}^{-n} & \varpi_{\mathrm{F}}^{-n-2}\mathrm{A}(\lambda) \\ 0 & \varpi_{\mathrm{F}}^{n} \end{pmatrix} \in \begin{cases} \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})\alpha_0^{n+2}\mathrm{K}_0 & \text{if } \mathrm{A}(\lambda) \in \mathcal{O}_{\mathrm{F}}^{\times} \\ \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})\alpha_0^{n+1}\mathrm{K}_0 & \text{if } \mathrm{A}(\lambda) \in \mathfrak{p}_{\mathrm{F}} \setminus \mathfrak{p}_{\mathrm{F}}^2 \\ \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})\alpha_0^{n}\mathrm{K}_0 & \text{if } \mathrm{A}(\lambda) \in \mathfrak{p}_{\mathrm{F}}^2 \end{cases}.$$

Hence, we have $\tau(\mathrm{R}_n^-(\sigma_{\vec{r}})) \subseteq \mathrm{R}_{n-1}^-(\sigma_{\vec{r}}) \oplus \mathrm{R}_n^-(\sigma_{\vec{r}}) \oplus \mathrm{R}_{n+1}^-(\sigma_{\vec{r}})$, for $n \geq 0$. □

Lemma 4.5. — *The map $\tau : \mathrm{C}_{0,\sigma_{\vec{r}}} \to \mathrm{C}_{1,\sigma_{\vec{r}}}$ is injective.*

Proof. Note that $\bar{\mathrm{U}}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})$-translates of $[\mathrm{I}, \mathrm{X}^{\vec{r}}]$ generates the space $\mathrm{C}_{0,\sigma_{\vec{r}}}$, by Proposition 2.1. Since the right hand side of the formula 2.4.1 has standard functions supported on distinct cosets, it suffices to show that for any non-zero $v \in \sigma_{\vec{r}}$ there exists some $\lambda \in k_{\mathrm{F}}$ such that

$$\mathrm{U}_{\vec{r}}\left(\sigma_{\vec{r}}\left(\begin{pmatrix} 0 & 1 \\ -1 & \mathrm{A}(\lambda) \end{pmatrix}\right)v\right) \neq 0.$$

As in the proof of Proposition 2.1 we consider $\mathrm{Sym}^{\vec{r}}\,\bar{\mathbf{F}}_p^2$ as an $\mathrm{SL}_2(k_{\mathrm{F}})$-subrepresentation of $\mathrm{Sym}^r\,\bar{\mathbf{F}}_p^2$ via the map $v_0 \otimes v_1 \otimes \cdots \otimes v_{n-1} \mapsto v_0 v_1^p \cdots v_{n-1}^{p^{n-1}}$ and work with the monomial basis

$$\left\{\mathrm{X}^{r-i}\mathrm{Y}^i \;:\; i = \sum_{j=0}^{n-1} i_j p^j \leq r = \sum_{j=0}^{n-1} r_j p^j,\; i_j, r_j \in \{0,\ldots,p-1\},\; i_j \leq r_j \text{ for every } j\right\}.$$

Now we write $v = \sum_i c_i \mathrm{X}^{r-i}\mathrm{Y}^i$ as a linear combination with the admissible monomials and coefficients $c_i \in \bar{\mathbf{F}}_p$ not all zero. For any $a \in k_{\mathrm{F}}$ the coefficient of X^r in $u(a)\cdot v$ is $\mathrm{P}_v(a) = \sum_i c_i a^i$ which is a polynomial (in a) of degree $r < q$ with coefficients $c_i \in \bar{\mathbf{F}}_p$ not all zero. Thus there exists some $a \in k_{\mathrm{F}}$ so that $\mathrm{P}_v(a) \neq 0$. We let $\mathrm{A}(\lambda) := [a]$ where $[\,\cdot\,]$ denotes the Teichmuller lift as usual. Therefore, the coefficient of X^r in $u(\mathrm{A}(\lambda))\cdot v$ is $\mathrm{P}_v(a)$ which is non-zero. Hence, the Y^r coefficient of

$$\begin{pmatrix} 0 & 1 \\ -1 & \mathrm{A}(\lambda) \end{pmatrix}\cdot v = w_0 u(\mathrm{A}(\lambda))\cdot v$$

is non-zero. So $\mathrm{U}_{\vec{r}}\left(\sigma_{\vec{r}}\left(\begin{pmatrix} 0 & 1 \\ -1 & \mathrm{A}(\lambda) \end{pmatrix}\right)v\right) \neq 0$ by eq. 2.4.2 as required. □

4.2. **Action of τ strictly increases the** top **value.** Next, we will show that τ satisfies the condition (C4). But we first need to explicitly compute the action of τ on $\mathrm{ind}_{\mathrm{K}_0}^{\mathrm{G}_{\mathrm{S}}}(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)}$ using the explicit formulae 2.4.1 and 2.5.1.

Lemma 4.6. — *We have*

(i) $\tau(f_0) = f_{-1} + \lambda_{\sigma_{\vec{r}}} f_1$, *where* $\lambda_{\sigma_{\vec{r}}} = \begin{cases} 0 & \textit{if } \vec{r} \neq \vec{0} \\ 1 & \textit{if } \vec{r} = \vec{0} \end{cases}$.

(ii) *For $n \neq 0$ we have $\tau(f_n) = c_n f_n + f_{n+\delta(n)}$, where c_n is a scalar and* $\delta(n) = \begin{cases} 1 & \textit{if } n > 0 \\ -1 & \textit{if } n < 0 \end{cases}$.

Proof. (i) At first, we consider the case when $\sigma_{\vec{r}} \neq 1$. Then $f_0 = [\mathrm{I}_2, \mathrm{X}^{\vec{r}}]$, and we have :

$$\begin{aligned}\tau(f_0) =& \tau([\mathrm{I}_2, \mathrm{X}^{\vec{r}}]) \\ =& \sum_{\lambda \in k_{\mathrm{F}}^2} \left[\begin{pmatrix} 1 & \mathrm{A}(\lambda) \\ 0 & 1 \end{pmatrix} \alpha_0^{-1}, w_0 \mathrm{U}_{\vec{r}} \sigma_{\vec{r}} \Big(\begin{pmatrix} 0 & 1 \\ -1 & \mathrm{A}(\lambda) \end{pmatrix} \Big) (\mathrm{X}^{\vec{r}}) \right] + \sum_{\mu \in k_{\mathrm{F}}} \left[\begin{pmatrix} 1 & 0 \\ \varpi_{\mathrm{F}} \mathrm{A}(\mu) & 1 \end{pmatrix} \alpha_0, \mathrm{U}_{\vec{r}} \mathrm{X}^{\vec{r}} \right] \\ =& \sum_{\lambda \in k_{\mathrm{F}}^2} \left[\begin{pmatrix} 1 & \mathrm{A}(\lambda) \\ 0 & 1 \end{pmatrix} \alpha_0^{-1}, w_0 \mathrm{U}_{\vec{r}} ((-1)^r \mathrm{Y}^{\vec{r}}) \right] \\ =& \sum_{\lambda \in k_{\mathrm{F}}^2} \left[\begin{pmatrix} 1 & \mathrm{A}(\lambda) \\ 0 & 1 \end{pmatrix} \alpha_0^{-1}, (-1)^r w_0 \cdot \mathrm{Y}^{\vec{r}} \right] \\ =& \sum_{\lambda \in k_{\mathrm{F}}^2} \left[\begin{pmatrix} 1 & \mathrm{A}(\lambda) \\ 0 & 1 \end{pmatrix} \alpha_0^{-1}, (-1)^{2r} \mathrm{X}^{\vec{r}} \right] \\ =& f_{-1}.\end{aligned}$$

Next, let $\sigma_{\vec{r}} = 1$. Then, we have :

$$\begin{aligned}\tau(f_0) =& \tau([\mathrm{I}_2,1]) \\ =& \sum_{\lambda \in k_{\mathrm{F}}^2} \begin{pmatrix} 1 & \mathrm{A}(\lambda) \\ 0 & 1 \end{pmatrix} \alpha_0^{-1} \cdot [\mathrm{I}_2,1] + \sum_{\mu \in k_{\mathrm{F}}} \begin{pmatrix} 1 & 0 \\ \varpi_{\mathrm{F}} \mathrm{A}(\mu) & 1 \end{pmatrix} \alpha_0 \cdot [\mathrm{I}_2,1] \\ =& \sum_{u \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^2)} [u\alpha_0^{-1},1] + \sum_{\bar{u} \in \bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^2)} [\bar{u}\alpha_0,1] \\ =& f_1 + f_{-1}.\end{aligned}$$

(ii) At first, we take $n = -m$ to be a negative integer. We will show that

$$\tau(f_n) = c_n f_n + f_{n-1}$$

for some $c_n \in \bar{\mathbf{F}}_p$. Now, f_{-m} is $\mathrm{I}_{\mathrm{S}}(1)$-invariant and as τ is a G_{S}-intertwiner, $\tau(f_{-m})$ is also $\mathrm{I}_{\mathrm{S}}(1)$-invariant. Since each $\mathrm{R}_j^+(\sigma_{\vec{r}})$ for $j \geq 0$ is $\mathrm{I}_{\mathrm{S}}(1)$-stable we have

$$\tau(f_{-m}) \in (\mathrm{R}_{m-1}^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_m^+(\sigma_{\vec{r}}) \oplus \mathrm{R}_{m+1}^+(\sigma_{\vec{r}}))^{\mathrm{I}_{\mathrm{S}}(1)} = \mathrm{R}_{m-1}^+(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)} \oplus \mathrm{R}_m^+(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)} \oplus \mathrm{R}_{m+1}^+(\sigma_{\vec{r}})^{\mathrm{I}_{\mathrm{S}}(1)},$$

and so we can write

$$\tau(f_{-m}) = c_{m-1} f_{-m+1} + c_m f_{-m} + c_{m+1} f_{-m-1}$$

for some $c_{m-1}, c_m, c_{m+1} \in \bar{\mathbf{F}}_p$. Now, by the formula 2.5.1 and part (1), we have

$$\tau(f_{-m}) = \tau\Big(\sum_{u \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m})} u\alpha_0^{-m} \cdot f_0 \Big) = \sum_{u \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m})} u\alpha_0^{-m} (f_{-1} + \lambda_{\vec{r}} f_1).$$

So, to find the constants c_{m-1} and c_{m+1} we evaluate $\tau(f_{-m})$ above at α_0^{m-1} and α_0^{m+1}. For this we at first need to find which K_0-$\mathrm{I}_{\mathrm{S}}(1)$ double coset the elements $\alpha_0^{m-1} u \alpha_0^{-m}$ and $\alpha_0^{m+1} u \alpha_0^{-m}$ lie in, for $u \in \mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m})$. Let u be an upper unipotent with the top right entry $x \in \mathcal{O}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{2m}$, and we have

$$\alpha_0^{m-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \alpha_0^{-m} = \begin{pmatrix} \varpi_{\mathrm{F}} & x\varpi_{\mathrm{F}}^{1-2m} \\ 0 & \varpi_{\mathrm{F}}^{-1} \end{pmatrix} \in \begin{cases} \mathrm{K}_0 \alpha_0^{-1} \mathrm{I}_{\mathrm{S}}(1) & \text{if } x \in \mathfrak{p}_{\mathrm{F}}^{2m-2} \\ \mathrm{K}_0 \alpha_0^{l(x)} \mathrm{I}_{\mathrm{S}}(1) & \text{if } x \in \mathcal{O}_{\mathrm{F}} \setminus \mathfrak{p}_{\mathrm{F}}^{2m-2} \end{cases},$$

where $l(x) < -1$; the above containments can be seen by elementary row and column reduction. For instance if $x = a\varpi_{\mathrm{F}}^{2m-2}$ with $a \in \mathcal{O}_{\mathrm{F}}$ then :

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi_{\mathrm{F}} & a\varpi_{\mathrm{F}}^{-1} \\ 0 & \varpi_{\mathrm{F}}^{-1} \end{pmatrix} = \alpha_0^{-1},$$

and if $x = a\varpi_{\mathrm{F}}^{2m-d} \in \mathcal{O}_{\mathrm{F}}$ with $d \geq 3$ and $a \in \mathcal{O}_{\mathrm{F}}^{\times}$ then :

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1}\varpi_{\mathrm{F}}^{d-2} & 1 \end{pmatrix} \begin{pmatrix} \varpi_{\mathrm{F}} & a\varpi_{\mathrm{F}}^{1-d} \\ 0 & \varpi_{\mathrm{F}}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1}\varpi_{\mathrm{F}}^{d} & 1 \end{pmatrix} = \alpha_0^{1-d}.$$

Then, we have

$$\begin{aligned}
\tau(f_{-m})(\alpha_0^{m-1}) = & \sum_{u \in \mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2m})} f_{-1}(\alpha_0^{m-1} u \alpha_0^{-m}) + \lambda_{\vec{r}} \sum_{u \in \mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2m})} f_1(\alpha_0^{m-1} u \alpha_0^{-m}) \\
= & \lambda_{\vec{r}} \sum_{u \in \mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2m-2})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2m})} f_1\left(\underbrace{\alpha_0^{m-1} u \alpha_0^{-m}}_{\text{write as } u_1\alpha_0^{-1} \text{ with } u_1 \in \mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2})} \right) \\
= & \lambda_{\vec{r}} \sum_{u_1 \in \mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2})} f_1(u_1 \alpha_0^{-1}) \\
= & \lambda_{\vec{r}} \sum_{\mu \in k_{\mathrm{F}}^2} \begin{pmatrix} 1 & [\mu_0]+[\mu_1]\varpi_{\mathrm{F}} \\ 0 & 1 \end{pmatrix} \cdot f_1(\alpha_0^{-1}) \\
= & \lambda_{\vec{r}} \sum_{\mu \in k_{\mathrm{F}}^2} \begin{pmatrix} 1 & [\mu_0]+[\mu_1]\varpi_{\mathrm{F}} \\ 0 & 1 \end{pmatrix} \cdot (w_0 \cdot v_{\sigma_{\vec{r}}}) \\
= & \lambda_{\vec{r}} \sum_{(\mu_0,\mu_1) \in k_{\mathrm{F}} \times k_{\mathrm{F}}} \begin{pmatrix} 1 & [\mu_0] \\ 0 & 1 \end{pmatrix} \cdot (w_0 \cdot v_{\sigma_{\vec{r}}}) \\
= & 0.
\end{aligned}$$

Hence, $c_{m-1} = 0$. Next, we take u to be an upper unipotent with the top right entry $x \in \mathcal{O}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{2m}$, and we have

$$\alpha_0^{m+1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \alpha_0^{-m} = \begin{pmatrix} \varpi_{\mathrm{F}}^{-1} & x\varpi_{\mathrm{F}}^{-2m-1} \\ 0 & \varpi_{\mathrm{F}} \end{pmatrix} \in \begin{cases} \mathrm{K}_0\alpha_0\mathrm{I_S}(1) & \text{if } x \in \mathfrak{p}_{\mathrm{F}}^{2m} \\ \mathrm{K}_0\alpha_0^{l'(x)}\mathrm{I_S}(1) & \text{if } x \in \mathcal{O}_{\mathrm{F}} \setminus \mathfrak{p}_{\mathrm{F}}^{2m} \end{cases},$$

where $l'(x) < -1$. The above containments can be proved as before. Therefore, we have :

$$\begin{aligned}
\tau(f_{-m})(\alpha_0^{m+1}) = & \sum_{u \in \mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2m})} f_{-1}(\alpha_0^{m+1} u \alpha_0^{-m}) + \lambda_{\vec{r}} \sum_{u \in \mathrm{U_S}(\mathcal{O}_{\mathrm{F}})/\mathrm{U_S}(\mathfrak{p}_{\mathrm{F}}^{2m})} f_1(\alpha_0^{m+1} u \alpha_0^{-m}) \\
= & f_{-1}(\alpha_0) = v_{\sigma_{\vec{r}}}.
\end{aligned}$$

Hence, $c_{m+1} = 1$.

Next, we consider the case when $n = m+1$ is positive i.e. $m \geq 0$. Then, $f_n = f_{m+1} \in \mathrm{R}_m^-(\sigma_{\vec{r}})$, hence

$$\tau(f_{m+1}) = c_{m-1} f_m + c_{m+1} f_{m+1} + c_{m+2} f_{m+2},$$

and we want to show that $c_{m-1} = 0$ and $c_{m+2} = 1$. Now, by the formula 2.5.1 and part (1), we get

$$\begin{aligned}\tau(f_{m+1}) =& \tau\Big(\sum_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})}[\bar{u}\alpha_0^{m+1}, w_0\cdot v_{\sigma_{\vec{r}}}]\Big) = \sum_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})}\bar{u}\alpha_0^{m+1}w_0\tau(f_0)\\ =& \sum_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})}\bar{u}\alpha_0^{m+1}w_0(f_{-1}+\lambda_{\sigma_{\vec{r}}}f_1)\end{aligned}$$

So, to find the constants c_{m-1} and c_{m+2} we evaluate $\tau(f_{m+1})$ at α_0^{-m} and α_0^{-m-2} respectively. As before, we at first find which K_0-$\mathrm{I}_{\mathrm{S}}(1)$ double coset the elements $\alpha_0^m u\alpha_0^{-m-1}$ and $\alpha_0^{m+2}u\alpha_0^{-m-1}$ lie in, for $u\in\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})$. Let u be an upper unipotent with the top right entry $x\in\mathfrak{p}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{2m+2}$. Then, we have

$$\alpha_0^m\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\alpha_0^{-m-1} = \begin{pmatrix}\varpi_{\mathrm{F}} & x\varpi_{\mathrm{F}}^{-2m-1}\\ 0 & \varpi_{\mathrm{F}}^{-1}\end{pmatrix}\in\begin{cases}\mathrm{K}_0\alpha_0^{-1}\mathrm{I}_{\mathrm{S}}(1) & \text{if } x\in\mathfrak{p}_{\mathrm{F}}^{2m}\\ \mathrm{K}_0\alpha_0^{l(x)}\mathrm{I}_{\mathrm{S}}(1) & \text{if } x\in\mathfrak{p}_{\mathrm{F}}\setminus\mathfrak{p}_{\mathrm{F}}^{2m}\end{cases},$$

where $l(x) < -1$. As a result we have

$$\begin{aligned}\tau(f_{m+1})(\alpha_0^{-m}) =& \sum_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})} f_{-1}(\alpha_0^{-m}\bar{u}\alpha_0^{m+1}w_0) + \lambda_{\vec{r}}\sum_{\bar{u}\in\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}})/\bar{\mathrm{U}}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})} f_1(\alpha_0^{-m}\bar{u}\alpha_0^{m+1}w_0)\\ =&\lambda_{\vec{r}}\sum_{u\in\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^{2m+2})} w_0\cdot f_1\Bigg(\underbrace{\alpha_0^m u\alpha_0^{-m-1}}_{\text{write as } u_1\alpha_0^{-1} \text{ with } u_1\in\mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^2)}\Bigg)\\ =&\lambda_{\vec{r}}\sum_{u_1\in\mathrm{U}_{\mathrm{S}}(\mathcal{O}_{\mathrm{F}})/\mathrm{U}_{\mathrm{S}}(\mathfrak{p}_{\mathrm{F}}^2)}(w_0u_1w_0)\cdot v_{\sigma_{\vec{r}}}\\ =&\lambda_{\vec{r}}\sum_{(\mu_0,\mu_1)\in k_{\mathrm{F}}^2}(w_0\begin{pmatrix}1 & [\mu_0]\\ 0 & 1\end{pmatrix}w_0)\cdot v_{\sigma_{\vec{r}}} = 0,\end{aligned}$$

and hence $c_{m-1} = 0$. Next, for u an upper unipotent with the top right entry $x\in\mathfrak{p}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{2m+2}$, we have

$$\alpha_0^{m+2}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\alpha_0^{-m-1} = \begin{pmatrix}\varpi_{\mathrm{F}}^{-1} & x\varpi_{\mathrm{F}}^{-2m-3}\\ 0 & \varpi_{\mathrm{F}}\end{pmatrix}\in\begin{cases}\mathrm{K}_0\alpha_0\mathrm{I}_{\mathrm{S}}(1) & \text{if } x\in\mathfrak{p}_{\mathrm{F}}^{2m+2}\\ \mathrm{K}_0\alpha_0^{l'(x)}\mathrm{I}_{\mathrm{S}}(1) & \text{if } x\in\mathfrak{p}_{\mathrm{F}}\setminus\mathfrak{p}_{\mathrm{F}}^{2m+2}\end{cases},$$

where $l'(x) < -1$. Hence, we get

$$\tau(f_{m+1})(\alpha_0^{-m-2}) = f_{-1}(\alpha_0^{-m-2}\alpha_0^{m+1}w_0) = w_0\cdot f_{-1}(\alpha_0) = w_0\cdot v_{\sigma_{\vec{r}}},$$

that is to say, $c_{m+2} = 1$ as required. This completes the proof. □

Finally we are ready to show that τ satisfies the condition (C4). Given the results we have established, the proof is a formal argument essentially same as [14, Lemma 4.4].

Lemma 4.7. — *For $n\geq 0$, let $f\in\mathrm{B}_{n+1,\sigma_{\vec{r}}}$. If $\tau(f)\in\mathrm{B}_{n+1,\sigma_{\vec{r}}}$ then $f\in\mathrm{B}_{n,\sigma_{\vec{r}}}$.*

Proof. Let $\mathrm{M}_{n+1,\sigma_{\vec{r}}}$ be the subspace of $\mathrm{B}_{n+1,\sigma_{\vec{r}}}$ consisting of functions f such that $\tau(f)\in\mathrm{B}_{n+1,\sigma_{\vec{r}}}$. So we have to show that $\mathrm{M}_{n+1,\sigma_{\vec{r}}}\subset\mathrm{B}_{n,\sigma_{\vec{r}}}$.

Suppose for contradiction that there exists some $f\in\mathrm{M}_{n+1,\sigma_{\vec{r}}}\setminus\mathrm{B}_{n,\sigma_{\vec{r}}}$. Then $f\in\mathrm{B}_{n+1,\sigma_{\vec{r}}} = \mathrm{B}_{n,\sigma_{\vec{r}}}\oplus\mathrm{C}_{n+1,\sigma_{\vec{r}}}$ and so we write $f = f' + f''$ for some $f'\in\mathrm{B}_{n,\sigma_{\vec{r}}}$, $f''\in\mathrm{C}_{n+1,\sigma_{\vec{r}}}$. Now note that $\mathrm{B}_{n,\sigma_{\vec{r}}}\subset\mathrm{M}_{n+1,\sigma_{\vec{r}}}$ since $\mathrm{B}_{n,\sigma_{\vec{r}}}\subset\mathrm{B}_{n+1,\sigma_{\vec{r}}}$ and $\tau(\mathrm{B}_{n,\sigma_{\vec{r}}})\subset\mathrm{B}_{n+1,\sigma_{\vec{r}}}$ by Lemma 4.4. Hence $f'' = f - f'\in\mathrm{M}_{n+1,\sigma_{\vec{r}}}$. But since $f\notin\mathrm{B}_{n,\sigma_{\vec{r}}}$ we have $f''\neq 0$. Thus $\mathrm{C}_{n+1,\sigma_{\vec{r}}}\cap\mathrm{M}_{n+1,\sigma_{\vec{r}}}\neq 0$.

By Remark 4.2 we know that $\mathrm{C}_{n+1,\sigma_{\vec{r}}}$ is $\mathrm{I_S}(1)$-stable for $n \geq 0$ and so $\mathrm{B}_{n+1,\sigma_{\vec{r}}}$ is $\mathrm{I_S}(1)$-stable for $n \geq 0$. Also since τ is $\mathrm{G_S}$-invariant $\mathrm{M}_{n+1,\sigma_{\vec{r}}}$ is $\mathrm{I_S}(1)$-stable. But since $\mathrm{I_S}(1)$ is a pro-p group there exists some non-zero vector $f^* \in \mathrm{C}_{n+1,\sigma_{\vec{r}}} \cap \mathrm{M}_{n+1,\sigma_{\vec{r}}}$ which is fixed by $\mathrm{I_S}(1)$. Now $\mathrm{C}_{n+1,\sigma_{\vec{r}}} := \mathrm{R}^+_{n+1}(\sigma_{\vec{r}}) \oplus \mathrm{R}^-_n(\sigma_{\vec{r}})$ and hence $\mathrm{C}^{\mathrm{I_S}(1)}_{n+1,\sigma_{\vec{r}}} = \mathrm{R}^+_{n+1}(\sigma_{\vec{r}})^{\mathrm{I_S}(1)} \oplus \mathrm{R}^-_n(\sigma_{\vec{r}})^{\mathrm{I_S}(1)} = \bar{\mathbf{F}}_p \cdot f_{-(n+1)} \oplus \bar{\mathbf{F}}_p \cdot f_{n+1}$ by Remark 4.2. We write $f^* = \lambda f_{-(n+1)} + \mu f_{n+1}$ and using Lemma 4.6(ii) we obtain that $\tau(f^*)$ is of the form

$$c_{-(n+1)} f_{-(n+1)} + d_{n+1} f_{n+1} + f_{-(n+2)} + f_{n+2}\,, \qquad \text{where } c_{-(n+1)}, d_{n+1} \text{ are scalars.}$$

Hence $\tau(f^*)$ does not lie in $\mathrm{B}_{n+1,\sigma_{\vec{r}}}$ because $f_{-(n+2)}, f_{n+2} \in \mathrm{C}_{n+2,\sigma_{\vec{r}}}$. Thus we have a contradiction. $\square$

We now complete the proof of our main theorem.

Proof of Theorem 4.3. To apply Lemma 3.1 we take $\mathrm{V} := \mathrm{ind}^{\mathrm{G_S}}_{\mathrm{K}_0}(\sigma_{\vec{r}})$ and $\mathrm{C}_n := \mathrm{C}_{n,\sigma_{\vec{r}}}$ for each $n \geq 0$. The operator $\mathrm{T} := \tau$. By Remark 4.1 and Lemmas 4.4, 4.5, 4.7 we have ensured that the conditions (C1)-(C4) are satisfied. Thus, as in Remark 3.2 we can choose non-empty subsets $\mathrm{A}_k \subset \mathrm{C}_k$ for each $k \geq 0$ such that $\bigcup_{n\geq 0} \bigsqcup_{i+j\leq n} \tau^i(\mathrm{A}_j)$ is a basis of $\mathrm{ind}^{\mathrm{G_S}}_{\mathrm{K}_0}(\sigma_{\vec{r}})$. Consequently, the set $\bigcup_{n\geq 0} \mathrm{A}_n$ forms a basis for the module action of $\mathcal{H}(\mathrm{G_S}, \mathrm{K}_0, \sigma_{\vec{r}}) = \bar{\mathbf{F}}_p[\tau]$ on $\mathrm{ind}^{\mathrm{G_S}}_{\mathrm{K}_0}(\sigma_{\vec{r}})$. $\square$

Remark 4.8. – The freeness result provides a transparent and conceptual understanding of the compactly induced representations of SL_2 in the mod-p setting. Beyond its intrinsic interest, the heart of the matter involves the verification of the hypotheses laid out in the formal framework of the Lemma 3.1, and this idea may be adapted to split reductive groups of higher ranks when the corresponding Hecke action exhibits a graded or triangular behavior. Note that it is essential to consider split groups as in this case the spherical Hecke algebra is known to be commutative (see [11, Corollary 1.3]). Such constructions may play a role in the study of supersingular representations.

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